

UNIT - III

## Random Processes - Temporal characteristics

### Conditional Distribution

Let  $x$  be a random variable. let  $P(x \leq x_1)$  be an event and  $y$  be another event.

Then, the probability  $P(x \leq x_1 | y)$  is named as the conditional distribution function of  $x$ , denoted by  $F_x(x_1 | y)$ .

It is defined as

$$F_x(x_1 | y) = P(x \leq x_1 | y) = \frac{P(x \leq x_1 | y)}{P(y)}$$

$P(x \leq x_1 | y)$  indicates the probability of joint event  $(x \leq x_1, y)$  i.e.  $P(x \leq x_1, y)$

Properties of Conditional Distribution Function.

1.  $F_x(-\infty | y) = 0$

$$F_x(-\infty | y) = P(x \leq -\infty | y)$$

since  $F_x(-\infty) = P(x \leq -\infty) = 0$

$$\therefore F_x(-\infty | y) = 0.$$

2.  $F_x(\infty | y) = 1$

$$F_x(\infty | y) = P(x \leq \infty | y)$$

since  $F_x(\infty) = P(x \leq \infty) = 1$

$$F_x(\infty | y) = 1$$

3.  $0 \leq F_x(x_1 | y) \leq 1$

$$F_x(x_1 | y) = P(x \leq x_1 | y)$$

Since CDF of a random variable is numerically.

bounded between 0 and 1 i.e., since CDF of a random  $0 \leq F_X(x) \leq 1$ ,  $P(X \leq x|y)$  is also bounded between same values.

4. If  $x_1 < x_2$ ,  $F_X(x_1|y) \leq F_X(x_2|y)$

Since  $F_X(x_1) \leq F_X(x_2)$  for  $x_1 < x_2$ , the same result applies for conditional distribution also.

5. If  $x_1 < x_2$ ,  $F_X(x_2|y) - F_X(x_1|y) = P(x_1 < X \leq x_2|y)$   
This can be concluded from the property of CDF i.e.  
 $P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1)$  for  $x_1 < x_2$ .

In defining the conditional distribution  $F_X(x|y) = P[X \leq x|y]$ , the event  $y$  can be defined in terms of  $x$  or in terms of some other random variables.

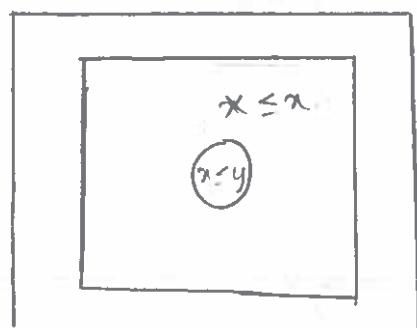
Let the event  $y$  be defined in terms of  $x$

$$\text{i.e. } y = (x \leq y)$$

$$\begin{aligned} \text{Then } F_X(x|y) &= F_X(x|y \leq y) = P[(x \leq x) \cap (x \leq y)] \\ &= \frac{P[(x \leq x) \cap (x \leq y)]}{P(x \leq y)} \end{aligned}$$

(a) If  $y \leq x$ , then  $P[(x \leq x) \cap (x \leq y)] = (P x \leq y)$ .

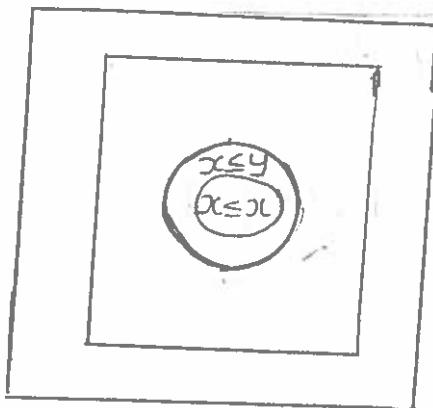
$$\therefore F_X(x|y) = F_X(x|x \leq y) = \frac{P(x \leq y)}{P(x \leq y)} = 1$$



b. If  $x < y$ , then  $P[(x \leq x) \cap (x \leq y)] = P(x \leq x)$ .

$$\therefore F_x(x|y) = F_x(x|x \leq y) = \frac{P(x \leq x)}{P(x \leq y)} = \frac{F_x(x)}{F_x(y)}$$

$$\therefore F_x(x|x \leq y) = \frac{F_x(x)}{F_x(y)} \quad \text{for } x < y$$
$$= 1 \quad \text{for } x \geq y.$$



### Conditional Density Function:-

This is defined on similar lines as the ordinary density function i.e., conditional density function is defined as the derivative of conditional distribution function i.e.,

$$f_x(x|y) = \frac{d}{dx} [F_x(x|y)]$$

$$\text{In the above case of } F_x(x|x \leq y) = \frac{F_x(x)}{F_x(y)} \quad \text{for } x < y$$

$$= 1 \quad \text{for } x \geq y.$$

The corresponding conditional density function is

$$f_x(x|x \leq y) = \frac{F_x(x)}{\int_{-\infty}^y f_x(x) dx} \quad \text{for } x < y$$

$$= 0 \quad \text{for } x \geq y.$$

## Properties of conditional Density Function:-

Conditional density also satisfies the properties same as that of ordinary density function.

1. It is always non negative i.e.  $f_x(x|y) \geq 0$ .

2. It encloses unit area i.e.,  $\int_{-\infty}^{\infty} f_x(x|y) dx = 1$

3.  $F_x(x|y) = \int_{-\infty}^x F_x(x|y) dx$ .

4.  $P(x_1 < x \leq x_2|y) = \int_{x_1}^{x_2} f_x(x|y) dx$ .

## Binomial Distribution:

Consider a random experiment having only two possible outcomes. Let the experiment be tossing a fair coin. Let the falling of head corresponds to success and falling of tail corresponds to failure. Let the probability of success be  $p$  and of the failure be  $q$ , such that  $p+q=1$ . If this experiment is repeated ' $n$ ' times independently with two possible outcomes, they are called Bernoulli Trials. In the above experiment, the probability of getting ' $m$ ' number of heads and  $(n-m)$  number of tails, in ' $n$ ' independent trials in a particular order i.e., HHTHHT...TH is

$$P(\text{HHTHHT...TH}) = P(H) \cdot P(T) \cdot P(T) \dots P(H).$$

$$= p \cdot q \cdot q \dots p = p \cdot p \dots q \cdot q \dots \\ m \text{ times } (n-m) \text{ times.}$$

Thus, the required probability is  $p^m \cdot (q)^{n-m}$ ;  $m=0, 1, \dots, n$ .

The number of ways to get 'm' number of heads and  $(n-m)$  number of tails in 'n' independent trials is  $nC_m$ . 3

Thus, the probability of 'm' successes in 'n' trials in any order is

$$P(X=m) = p(m) = nC_m \cdot p^m \cdot q^{n-m}, \quad m=0, 1, \dots, n.$$

$$= 0 \quad , \text{ otherwise.}$$

This is called Binomial distribution of order  $n$  and parameter  $p$ .

For various values of  $m (= 0, 1, \dots, n)$ , the above series is  $q^n, nC_1 \cdot p \cdot q^{n-1}, nC_2 \cdot p^2 \cdot q^{n-2}, \dots, p^n$ . These are the successive terms of the binomial expansion of  $(p+q)^n$ , and hence is the same Binomial distribution. A binomial variable follows Binomial distribution.

In general, in an experiment consisting of 'n' terms, if head occurs 'm' times, the binomial density function (probability) is given by (for 'm' no. of success).

$$P(X=m) = p(m) = nC_m \cdot p^m \cdot q^{n-m}, \quad m=0, 1, \dots, n.$$

$$= 0 \quad ; \text{ otherwise.}$$

Thus, the summation

$$q^n + nC_1 \cdot p \cdot q^{n-1} + nC_2 \cdot p^2 \cdot q^{n-2} + \dots + p^n$$

i.e.  $\sum_{m=0}^n nC_m \cdot p^m \cdot q^{n-m}$  is called the binomial density function.

$$\text{Thus } f(x) = \sum_{m=0}^n nC_m \cdot p^m \cdot q^{n-m} \cdot g(x-m).$$

The corresponding CDF i.e., Binomial distribution function is obtained by integrating the above pdf

$$\text{Thus } F(x) = \sum_{m=0}^n nC_m \cdot p^m \cdot q^{n-m} \cdot u(x-m).$$

$$\text{i.e. } F(x) = P(X \leq x) = \sum_{m=0}^x nC_m \cdot p^m \cdot q^{n-m}.$$

Thus Binomial distribution is used in the study of communication systems. In the transmission of digital information, in a message of  $n$  digits, ' $m$ ' stands for the number of errors in message.

Problem:-

The continuous random variable  $x$  has a pdf  $f(x) = \frac{x}{2}$  for  $0 \leq x \leq 2$ . Two independent determinations are made. What is the probability that both these determinations will be greater than one. If three independent determinations are made, what is the probability that exactly two of these are larger than one?

Let  $x_1$  and  $x_2$  be the determinations made.

$$P(\text{determination} > 1) = \int_{1}^2 \frac{x}{2} dx = \frac{x^2}{4} \Big|_1^2 = \frac{3}{4}.$$

Considering Binomial distribution,

Let determination  $> 1$  be the success. So,  $P = 3/4 = 2 = 1/4$

$$P(X=2) = 2C_2 \cdot (3/4)^2 \cdot (1/4)^0 = 9/16.$$

Let  $x_1, x_2$  and  $x_3$  be the determinations made.

$$P(X=2) = 3C_2 \cdot (3/4)^2 \cdot (1/4)^1 = 27/64.$$

## Uniform Distribution :-

This is also referred as Rectangular distribution.  
A random variable  $x$  is said to be a uniform random variable over an interval  $(a, b)$  if its probability density function is a constant 'K' over the the interval i.e.,

$$f_x(x) = K \quad \text{for } a \leq x \leq b.$$

$$= 0 \quad \text{elsewhere.}$$

Since, the area enclosed by any valid density function is unity.

$$\int_a^b f_x(x) \cdot dx = 1 \quad \text{i.e.,} \quad \int_a^b K \cdot dx = 1 \Rightarrow K = \frac{1}{b-a}.$$

$$\therefore f_x(x) = \frac{1}{b-a} \quad \text{for } a \leq x \leq b$$

$$= 0 \quad \text{elsewhere.}$$

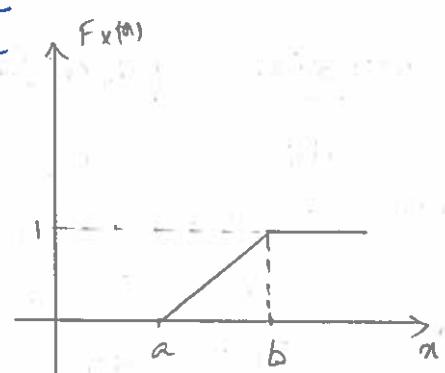
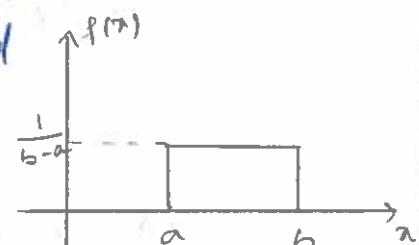
The above uniform density is plotted  
the corresponding cumulative distribution  
function is:

$$F_x(x) = \int_a^x f_x(n) \cdot dn = \int_a^x \frac{1}{b-a} \cdot dn = \frac{x-a}{b-a}$$

$$\therefore F_x(x) = 0 ; \text{ for } x < a$$

$$= \frac{x-a}{b-a} ; \text{ for } a \leq x \leq b$$

$$= 1 ; \text{ for } x > b.$$



The Uniform density finds its application in the conversion of analog and continuous information into digital form, where the process of quantization is involved. This process involves dividing the total peak-to-peak range into some discrete levels or level fli.

called quantization levels and each sample value of the signal will be rounded off to its nearest quantization level.

This rounding off results in an error, named as quantization error. Between its maximum and minimum values, quantization error is assumed as a uniform random variable.

**Problem:** A random variable is uniformly distributed between 1 and 3. Find the probability that the variable is in the range 1.5 and 2.

**Sol:** Let the given uniform density be

$$f_x(x) = k \quad \text{for } 1 \leq x \leq 3 \\ = 0 \quad \text{elsewhere.}$$

since  $\int_1^3 f_x(x) \cdot dx = 1$ ;  $\int_1^3 k \cdot dx = 1 \Rightarrow k = \frac{1}{2}$ .

$$\therefore f_x(x) = \frac{1}{2} \quad \text{for } 1 < x < 3$$

$$= 0 \quad \text{elsewhere.}$$

$$P(1.5 < x < 2) = \int_{1.5}^2 \frac{1}{2} dx = \frac{1}{2} [x]_{1.5}^2 = \frac{1}{4}.$$

### Gaussian Distribution:-

This is also referred to as Normal distribution. Most of the naturally occurring phenomenon are characterized by random variable distributed according to Gaussian density function.

A random variable 'x' is said to be Gaussian or Normal distributed, if its density function is

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where  $\mu$  = mean of the random variable.

$\sigma^2$  = Variance of the random variable. Unit-3, 8/16

The Gaussian / Normal distribution is denoted as  $N(m, \sigma^2)$ .

$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}$  is also the density of a Gaussian Random variable with zero mean and variance  $\sigma^2$ , denoted as  $N(0, \sigma^2)$ .

$f_x(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$  is the density of a Gaussian Random variable with zero mean and variance  $\sigma^2$ , denoted as  $N(0, \sigma^2)$ .

$f_x(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$  is the density of a Gaussian Random variable with zero mean and unity variance, denoted by  $N(0, 1)$ . This is referred to as standardized normal density.

Consider  $f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}$

Example:-

Solve the eqn  $\frac{d}{dx} f(x) = 0$

$$\Rightarrow \frac{d}{dx} \left[ \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} \right] = 0 \Rightarrow \frac{1}{\sqrt{2\pi}\sigma} \left[ -e^{-\frac{(x-m)^2}{2\sigma^2}} \cdot \frac{2(x-m)}{\sigma^2} \right] = 0$$

$$\Rightarrow x-m = 0 \Rightarrow x=0.$$

Thus, Gaussian density function has its maximum at  $x=m$ .

The maximum value is  $f(x) \Big|_{x=m} = \frac{1}{\sqrt{2\pi}\sigma}$

The Gaussian density function is a Bell shaped curve, having its peak at  $x=m$  and is symmetrical about its mean.

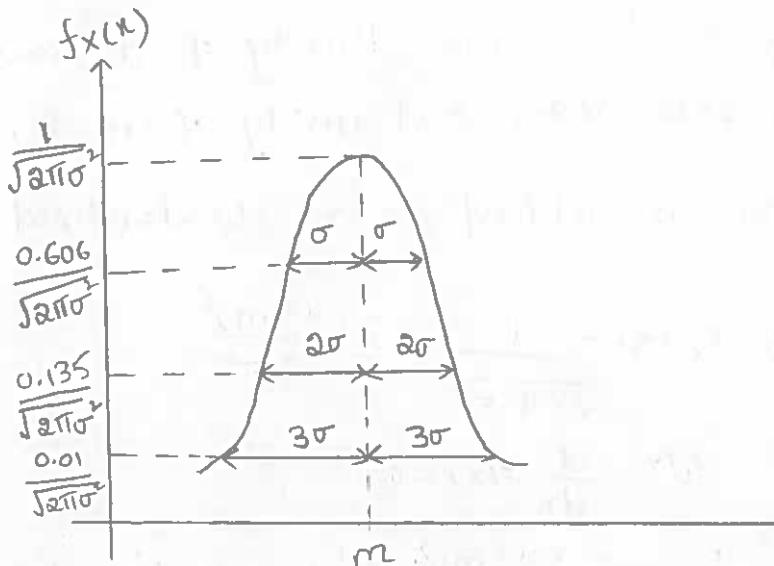
$$\text{when } x-m = \pm\sigma. \quad f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2}} = \frac{0.606}{\sqrt{2\pi}\sigma}.$$

i.e., at values of  $x$  separated from 'm' by  $\sigma$  ( $\sim$ ) is called standard deviation,  $f(x)$  be 0.606 times its peak value.

When  $x-m = \pm 2\sigma$   $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-2} = \frac{0.135}{\sqrt{2\pi\sigma^2}}$ , i.e., at values of  $x$  separated from 'm' by  $2\sigma$ ,  $f(x)$  will be 0.135 times its peak value.

$$\text{When } x-m = \pm 3\sigma \quad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-4.5} = \frac{0.01}{\sqrt{2\pi\sigma^2}}$$

i.e. at values of  $x$  separated from 'm' by  $3\sigma$ ,  $f(x)$  will be 0.01 times its peak value.



### Properties of Gaussian Density Function:-

1. The normal curve is a bell shaped curve and symmetrical about its mean.
2. The area enclosed by the Gaussian density curve is unity, i.e.,  $\int_{-\infty}^{\infty} f(x) dx = 1$

Consider  $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} dx$

let  $t = \frac{x-m}{\sigma} \Rightarrow dt = \frac{dx}{\sigma} \Rightarrow dx = \sigma \cdot dt$

$$\int_{-\infty}^{\infty} f(x) \cdot dx = \int_{-\infty}^{\infty} e^{-t^2/2} \cdot \frac{\sigma \cdot dt}{\sqrt{2\pi\sigma^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

let  $I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt$

$$\text{Consider } I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2}} dr \cdot \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(s^2+r^2)}{2}} ds dr$$

Let  $r = \sqrt{x^2 + y^2}$ ;  $\theta = \tan^{-1} \frac{y}{x}$ ;  $dr ds = r dr d\theta$   
 for  $0 \leq \theta \leq 2\pi$ ,  $0 \leq r \leq \infty$

This is referred to as polar co-ordinate system.

$$\therefore I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} r e^{-r^2/2} dr d\theta.$$

$$\text{Consider } \int_0^{\infty} r e^{-r^2/2} dr.$$

$$\text{Let } \frac{r^2}{2} = t \Rightarrow r dr = dt \quad \therefore \int_0^{\infty} r e^{-r^2/2} dr = \int_0^{\infty} e^{-t} dt = 1$$

$$\therefore I^2 = \frac{1}{2\pi} \int_0^{2\pi} 1 d\theta = 1 \Rightarrow I = 1 \quad \therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx = 1$$

3. The average or mean deviation of Gaussian distribution from its mean ' $m$ ' is given as

$$\int_{-\infty}^{\infty} |x-m| \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

$$\text{Let } \frac{x-m}{\sigma} = p \Rightarrow dx = \sigma dp.$$

Thus, the above integration becomes

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} -|p| \cdot e^{-p^2/2} \cdot \sigma \cdot dp = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |p| e^{-p^2/2} dp.$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[ \int_{-\infty}^0 -p e^{-p^2/2} dp + \int_0^{\infty} p e^{-p^2/2} dp \right] = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} p e^{-p^2/2} dp.$$

$$\text{Let } \frac{p^2}{2} = t \Rightarrow \frac{2p \cdot dp}{2} = dt \Rightarrow p dp = dt.$$

∴ The above integral is written as

$$\frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} e^{-t} dt = \frac{2\sigma}{\sqrt{2\pi}} = 0.7977 \sigma \approx 0.8 \sigma.$$

∴ Thus Mean distribution of Gaussian density function from its mean value ∴ n.e. ... 11/11

Problem :-

An analog signal received at the detector (measured in mV) may be modeled as a Gaussian Random variable  $N(200, 256)$  at a fixed point in time. What is the probability that the signal will exceed 240 mV. what is the probability that the signal is larger than 240 mV, given that it is larger than 210 mV?

Sol:

Let  $X$  be gaussian Random variable with density  $N(m, \sigma^2) = N(200, 256)$ .

$$f(x) = \frac{1}{\sqrt{2\pi(256)}} e^{-\frac{(x-200)^2}{2(256)}}$$

$$P(X > 240) = \int_{240}^{\infty} \frac{1}{\sqrt{2\pi(256)}} e^{-\frac{(x-200)^2}{2(256)}} dx.$$

$$\text{Let } \frac{x-200}{\sqrt{256}} = t \Rightarrow dx = \sqrt{256} dt$$

$$\text{Lower Limit for } t = \frac{240 - 200}{\sqrt{256}} = \frac{40}{16} = 2.5$$

$$\text{Upper Limit for } t = \infty$$

$$P(X > 240) = \int_{2.5}^{\infty} \frac{1}{\sqrt{2\pi(256)}} e^{-\frac{t^2}{2}} \sqrt{256} dt = \frac{1}{\sqrt{2\pi}\sqrt{256}} \int_{2.5}^{\infty} e^{-\frac{t^2}{2}} dt.$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{t^2}{2}} dt - \frac{1}{\sqrt{2\pi}} \int_0^{2.5} e^{-\frac{t^2}{2}} dt$$

$$\text{From tables } P(X > 240) = 0.5 - 0.4938 = 6.2 \times 10^{-3}$$

$$\text{Now consider } P[X > 240 | X > 210] = \frac{P[X > 240] + P[X > 210]}{P(X > 210)}$$

$$= \frac{P(X > 240)}{P(X > 210)}$$

$$P(X > 210) = \int_{210}^{\infty} \frac{1}{\sqrt{2\pi(256)}} e^{-\frac{(x-200)^2}{2(256)}} dx$$

$$\text{Let } \frac{x-200}{\sqrt{256}} = t \Rightarrow dx = \sqrt{256} dt$$

Lower Limit for  $t = \frac{10}{\sqrt{256}} = 0.625$

Upper limit for  $t = \infty$

$$P(X > 210) = \frac{1}{\sqrt{2\pi}} \int_{0.625}^{\infty} e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2} dt - \frac{1}{\sqrt{2\pi}} \int_0^{0.625} e^{-t^2/2} dt.$$

From tables,  $= 0.5 - 0.2357 = 0.2643$

$$\therefore P(X > 240 | X > 210) = \frac{P(X > 240)}{P(X > 210)} = \frac{6.2 \times 10^{-3}}{0.2643} = 0.023.$$

### Rayleigh Density Function:-

This density function holds a special relation to Gaussian density.

Consider two Gaussian Random variables  $x$  and  $y$  with zero mean and same variance  $\sigma^2$ .

$$\text{Thus, } f_x(\eta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\eta^2/2\sigma^2} \text{ and } f_y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}.$$

Let  $x = r\cos\theta$ ,  $y = r\sin\theta$  for  $0 \leq \theta \leq 2\pi$  and  $0 \leq r \leq \infty$ .  
Then  $dr dy = r dr d\theta$ .

We have started with two random variables  $x$  and  $y$  and two new random variables  $r$  and  $\theta$  are introduced.

Their joint density  $f(r, \theta)$  is obtained as follows.

About the joint density function of two random variables, refer.

$$f(r, \theta) = f(r, y) |\bar{J}(r, y)|$$

$$\text{where } \bar{J}(r, y) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r[\cos^2\theta + \sin^2\theta] = r.$$

$$\therefore f(r, \theta) = r f(r, y).$$

Let  $x$  and  $y$  be two independent random variables.

$$\begin{aligned} \therefore f(r, \theta) &= r f_X(x) f_Y(y) = r \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \\ &= \frac{r}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} \quad \text{for } 0 \leq r < \infty; 0 \leq \theta < 2\pi. \end{aligned}$$

$$\text{since } x^2 + y^2 = r^2.$$

$$f(r, \theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \quad \text{for } 0 \leq r \leq \infty$$

$$0 \leq \theta \leq 2\pi$$

$$\begin{aligned} f(r) &= \int_0^{2\pi} f(r, \theta) d\theta = \int_0^{2\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} d\theta \\ &= \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \quad \text{for } 0 \leq r \leq \infty. \end{aligned}$$

This  $f(r)$  is called Rayleigh density function.

Thus, it is defined as

$$\begin{aligned} f(r) &= \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \quad \text{for } r \geq 0 \\ &= 0 \quad \text{elsewhere.} \end{aligned}$$

Now solve the equation  $\frac{d}{dr} f(r) = 0$

$$\text{i.e. } \frac{d}{dr} \left[ \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \right] = 0$$

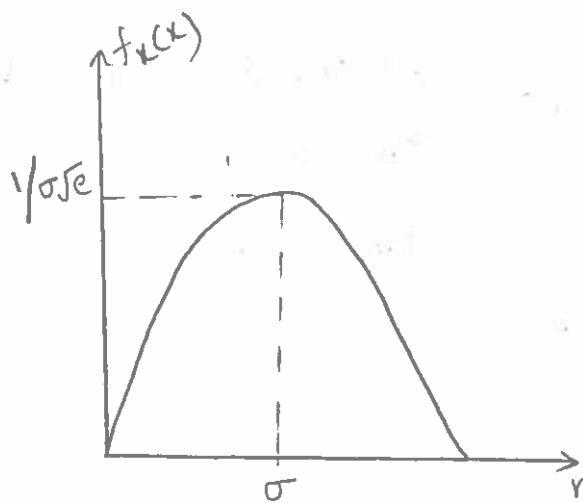
$$\Rightarrow \frac{1}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} - \frac{r}{\sigma^2} \cdot \frac{2r}{2\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = 0 \Rightarrow \frac{1}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} - \frac{r^2}{\sigma^4} e^{-\frac{r^2}{2\sigma^2}} = 0$$

$$\Rightarrow \sigma^2 e^{-\frac{r^2}{2\sigma^2}} - r^2 e^{-\frac{r^2}{2\sigma^2}} = 0 \Rightarrow \sigma^2 - r^2 = 0$$

$$\Rightarrow \sigma = r.$$

So, Rayleigh density function is having its peak value at  $r = \sigma$  and peak value is

$$f(r) \Big|_{r=\sigma} = \sigma \cdot e^{-\frac{\sigma^2}{2\sigma^2}} \text{ i.e., } \frac{1}{\sigma\sqrt{e}}.$$



The Rayleigh density function is plotted as  
Thus, Rayleigh density function is defined only for  
non negative  $r$ .

The corresponding CDF is

$$F(r) = \int_0^r f_X(x) dx = \int_0^r \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx.$$

$$\text{Let } \frac{x^2}{2\sigma^2} = t \Rightarrow \frac{2x dx}{2\sigma^2} = dt \Rightarrow \frac{x dx}{\sigma^2} = dt$$

Lower limit for  $t = 0$

Upper limit for  $t = \frac{r^2}{2\sigma^2}$

$$\therefore F(r) = \int_0^{\frac{r^2}{2\sigma^2}} e^{-t} dt = -e^{-t} \Big|_0^{\frac{r^2}{2\sigma^2}} = 1 - e^{-r^2/2\sigma^2}.$$

$$\therefore F(r) = 0 \quad \text{for } r < 0$$

$$= 1 - e^{-r^2/2\sigma^2} \quad \text{for } r > 0$$

Rayleigh density function is used to represent  
the envelope of narrow band noise, which is used  
in the noise analysis of various communication  
systems.

Problem:

A Rayleigh density function is given by

$$f_X(x) = \pi \cdot e^{-x^2/2} \quad \text{for } x \geq 0 \\ = 0 \quad \text{for } x < 0$$

Find the CDF

$$\text{Sol: } F_X(x) = \int_0^x f(t) dt = \int_0^x \pi e^{-t^2/2} dt$$

$$\text{let } t^2/2 = P \Rightarrow \frac{2t}{2} dt = dp \Rightarrow t dt = dp$$

Lower Limit for p=0

Upper Limit for p=x^2/2

$$\therefore F_X(x) = \int_x^{x^2/2} -e^{-P} dp = -e^{-P} \Big|_x^{x^2/2} = 1 - e^{-x^2/2}$$

$$\therefore F_X(x) = [1 - e^{-x^2/2}] u(x)$$